

Dark energy with polytropic equation of state

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(Dated: February 5, 2008)

Equation of state parameter plays a significant role for guessing the real nature of dark energy. In the present paper polytropic equation of state $p = \omega \rho^n$ is chosen for some of the kinematical Λ -models viz., $\Lambda \sim (\dot{a}/a)^2$, $\Lambda \sim \ddot{a}/a$ and $\Lambda \sim \rho$. Although in dust cases ($\omega = 0$) closed form solutions show no dependency on the polytropic index n , but in non-dust situations some new possibilities are opened up including phantom energy with supernegative ($\omega < -1$) equation of state parameter.

PACS numbers: 04.20.-q, 04.20.Jb, 98.80.Jk

I. INTRODUCTION

Ever since the discovery of an accelerating Universe through SN Ia observations [1, 2], scientists are searching for the cause behind this acceleration. It has been suspected that some kind of yet unknown energy is responsible for injecting the right amount of energy to the Universe for changing it to an accelerating one from a decelerating phase. Scientific community has coined the name *dark energy* for this unknown energy. Various types of models have been proposed for approximating this dark energy so far [3, 4].

It is observed that equation of state parameter plays a crucial role in understanding the actual nature of dark energy [5]. Till now, most of the models of Λ (the so-called cosmological constant of Einstein and a representative symbol for dark energy) with dynamic character have relied on the barotropic equation of state with wide range of values of the equation of state parameter ω , viz., $\omega = 0$ (for dust filled Universe), $\omega = 1/3$ (for radiation), $\omega = -1$ (for vacuum), $-1 < \omega < 0$ (for quintessence) and $\omega < -1$ (for phantom energy). But, all these models are plagued by some shortcomings. For instance, models with $\omega = 0$ and $1/3$ show an excess [6] or very low [3] age of the Universe while the stiff-fluid model, in spite of its nice agreement with the present age of the Universe [7], is not, in general, accepted as the real nature of the present Universe. In this circumstances, it is not unreasonable to think of an equation of state different from the barotropic one. This has prompted us to investigate the reaction of some of Λ models when polytropic equation of state is chosen.

Polytropic equation of state has been used in various astrophysical situations such as in the case of Lane-Emden models [8, 9]. In the present investigation it is used in cosmological realm. As test models we have selected $\Lambda \sim (\dot{a}/a)^2$, $\Lambda \sim \ddot{a}/a$ and $\Lambda \sim \rho$ models – the same three kinematical Λ -models which were chosen in one of our previous works [10] under the barotropic

equation of state. Here Sections 2 and 3 deal with the field equations and their solutions under three different Λ -models while various physical implications of the present work are discussed in Section 4.

II. EINSTEIN FIELD EQUATIONS

The Einstein field equations are

$$R^{ij} - \frac{1}{2}Rg^{ij} = -8\pi G \left[T^{ij} - \frac{\Lambda}{8\pi G}g^{ij} \right] \quad (1)$$

where *cosmological constant* Λ is assumed as a function of time, viz., $\Lambda = \Lambda(t)$ and the velocity of light c in vacuum is unity when expressed in relativistic units.

For the spherically symmetric Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (2)$$

where a is the scale factor and k , the curvature constant $-1, 0, +1$ respectively for open, flat and close models of the Universe, the Einstein field equations (1) take the forms as

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \quad (3)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \quad (4)$$

As various observational results [11] and inflation theory indicate that the Universe is flat so we have assumed here $k = 0$.

Let us choose the polytropic equation of state

$$p = \omega \rho^n \quad (5)$$

where equation of state parameter w can take the constant values 0, $1/3$, -1 and $+1$ respectively for the dust, radiation, vacuum fluid and stiff fluid and n is the polytropic index.

III. PARTICULAR SOLUTIONS

$$\mathbf{A.} \quad \Lambda \sim (\dot{a}/a)^2$$

If we use the *ansatz*

$$\Lambda = 3\alpha \left(\frac{\dot{a}}{a} \right)^2 \quad (6)$$

where α is a constant, then using equation (6) we get from equation (3)

$$3 \left(\frac{\dot{a}}{a} \right)^2 - 3\alpha \left(\frac{\dot{a}}{a} \right)^2 = 8\pi G\rho \quad (7)$$

which takes the form for density as follows:

$$\rho = \frac{3(1-\alpha)}{8\pi G} \left(\frac{\dot{a}}{a} \right)^2. \quad (8)$$

Using equation (5) and equation (6) we get from equation (4)

$$3 \left(\frac{\ddot{a}}{a} \right) = -12\pi G\omega\rho^n - 4\pi G\rho + 3\alpha \left(\frac{\dot{a}}{a} \right)^2 \quad (9)$$

which, on simplification, yields

$$a^{2n-1} \frac{d^2 a}{dt^2} + A \left(\frac{da}{dt} \right)^{2n} + B a^{2n-2} \left(\frac{da}{dt} \right)^2 = 0 \quad (10)$$

where $A = [3^n \omega (1-\alpha)^n] / [2^n (4\pi G)^{n-1}]$ and $B = (1-3\alpha)/2$.

Let us substitute, $z = da/dt$ for which the equation (10) reduces to

$$z^{1-2n} \frac{dz}{da} + A a^{1-2n} + z^{2-2n} \frac{B}{a} = 0. \quad (11)$$

If we put, $z^{2-2n} = y$, then differentiating it with respect to the scale factor a , we get

$$z^{1-2n} \frac{dz}{da} = \frac{1}{(2-2n)} \frac{dy}{da}. \quad (12)$$

Hence equation (11), by use of equation (12) becomes,

$$\frac{dy}{da} + \frac{(2-2n)B}{a} y = -(2-2n)A a^{1-2n}. \quad (13)$$

Now let us solve equation (13) in some particular cases for getting insight into the physical situations.

1. Dust case ($\omega = 0$)

For pressureless dust, $\omega = 0 = A$ and equation (13) becomes,

$$\frac{dy}{da} + \frac{(2-2n)B}{a} y = 0. \quad (14)$$

Solving equation (14) we get,

$$y = C_1 a^{(2n-2)B} \quad (15)$$

C_1 being an integration constant.

This immediately yields

$$\frac{da}{dt} = C_2 a^{-B} \quad (16)$$

where $C_2 = C_1^{1/(2-2n)}$.

Integrating equation (16) and using the initial condition that $a = 0$ for $t = 0$ we get,

$$a(t) = C_3 \left[\frac{3(1-\alpha)}{2} \right]^{2/3(1-\alpha)} t^{2/3(1-\alpha)} \quad (17)$$

where $C_3 = C_2^{1/(B+1)}$ being an integration constant, which provides the Hubble parameter as

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3(1-\alpha)} \frac{1}{t}. \quad (18)$$

Therefore from (8), we get

$$\rho(t) = \frac{1}{6\pi G(1-\alpha)} \frac{1}{t^2}. \quad (19)$$

Also, from equation (6), we obtain

$$\Lambda(t) = \frac{4\alpha}{3(1-\alpha)^2} \frac{1}{t^2}. \quad (20)$$

Thus, it is interesting to note that in dust case, expressions for $\rho(t)$ and $\Lambda(t)$ as well as t dependent part of $a(t)$ are independent of n . Moreover, $a(t)$, $\rho(t)$ and $\Lambda(t)$ follow the same power law with t as those in the dust case with barotropic equation of state (i.e. $n = 1$) [10]. But, surprisingly, equation (17) becomes undefined for $n = 1$ since C_3 contains the factor $(2-2n)$ in the denominator.

2. Non-dust case ($\omega \neq 0$)

When $\omega \neq 0$, then $A \neq 0$ and hence equation (11) is a linear equation. Multiplying equation (11) by integrating factor $a^{(2-2n)B}$ and solving the resulting equation we get,

$$y = -\frac{A a^{(2-2n)B}}{B+1} \quad (21)$$

which yields after substituting the value of y

$$\frac{da}{dt} = -\frac{A}{(B+1)}a. \quad (22)$$

By solving equation (22) we get our solution for the scale factor in the form

$$a(t) = C_4 \exp[(-\omega)^{1/2(1-n)} \frac{8\pi G}{3(1-\alpha)}]^{1/2} t \quad (23)$$

where C_4 is integration constant.

It is clear from equation (23) that no real value of $a(t)$ is possible if ω is positive. So, ω must be negative. Putting specifically, $\omega = -1$ in equation (23) we obtain

$$a(t) = C_4 \exp[8\pi G/3(1-\alpha)]^{1/2} t. \quad (24)$$

Now, $\omega = -1$ means a vacuum fluid. So, for vacuum fluid, we get an exponential solution independent of n . But, if $\omega < -1$, then scale factor depends on n . Now, $\omega < -1$ corresponds to a supernegative equation of state and hence the idea of phantom energy comes into the picture. Also, during inflation, the Universe underwent an exponential expansion. So, for non-dust case our solution reflects an inflationary scenario which may be due to either vacuum energy or quintessence or phantom energy. Also, from equation (24) we get,

$$H = \left[\frac{8\pi G}{3(1-\alpha)} \right]^{1/2}. \quad (25)$$

From equation (8) and (6) we get respectively

$$\rho(t) = 1, \quad (26)$$

$$\Lambda(t) = \frac{8\pi G\alpha}{1-\alpha}. \quad (27)$$

Therefore, the cosmic matter and vacuum energy densities are given by

$$\Omega_m = \frac{8\pi G\rho}{3H^2} = 1 - \alpha, \quad (28)$$

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} = \alpha. \quad (29)$$

So that, equations (29) and (30), immediately provide

$$\Omega_m + \Omega_\Lambda = 1. \quad (30)$$

Thus, we find that although polytropic equation of state yields interesting result by invoking the idea of phantom energy for non-dust case, yet it presents us some unacceptable situations like constant H (equation (25)) and constant ρ (equation (26)). A possible explanation of this will be discussed afterwards. However, the cosmic matter and vacuum energy density parameters satisfy the well-known relation $\Omega_m + \Omega_\Lambda = 1$ for flat Universe ($\Omega_k = 0$).

B. $\Lambda \sim \rho$

For this model we use the *ansatz*

$$\Lambda = 8\pi G\gamma\rho \quad (31)$$

where γ is a constant.

Then, from equation (3) we get

$$\rho = \frac{3}{8\pi G(\gamma+1)} \left(\frac{\dot{a}}{a} \right)^2. \quad (32)$$

Using (5) we get from (4)

$$3 \left(\frac{\ddot{a}}{a} \right) = -12\pi G\omega\rho^n - 4\pi G\rho + 8\pi G\gamma\rho \quad (33)$$

which, on simplification, yields the differential equation

$$a^{(2n-1)} \frac{d^2 a}{dt^2} + D \left(\frac{da}{dt} \right)^{2n} - E a^{(2n-2)} \left(\frac{da}{dt} \right)^2 = 0 \quad (34)$$

where $D = \omega 3^n / [2^n(\gamma+1)^n (4\pi G)^{n-1}]$ and $E = (2\gamma - 1)/2(\gamma+1)$.

Substituting $da/dt = u$ and then $v = u^{2-2n}$ equation (34) becomes

$$\frac{1}{(2-2n)} \frac{dv}{da} - \frac{E}{a} v = -D a^{1-2n}. \quad (35)$$

1. Dust case ($\omega = 0$)

For pressureless dust case $D=0$, so that equation (35) reduces to

$$\frac{dv}{da} + (2n-2) \frac{E}{a} v = 0. \quad (36)$$

Solving equation (36) in the same manner as in the previous model, we get our solution set as

$$a(t) = C_5 \left[\frac{3}{2(\gamma+1)} \right]^{2(\gamma+1)/3} t^{2(\gamma+1)/3} \quad (37)$$

where C_5 is a constant.

$$H(t) = \frac{2(\gamma+1)}{3} \frac{1}{t}, \quad (38)$$

$$\rho(t) = \frac{(\gamma+1)}{6\pi G} \frac{1}{t^2}, \quad (39)$$

$$\Lambda(t) = \frac{4\gamma(\gamma+1)}{3} \frac{1}{t^2}. \quad (40)$$

Also, we can calculate for the cosmic and vacuum energy densities which, respectively, are $\Omega_m = 1/(\gamma+1)$ and $\Omega_\Lambda = \gamma/(\gamma+1)$. Therefore, $\Omega_m + \Omega_\Lambda = 1$ and hence $\gamma = \Omega_\Lambda/\Omega_m$. Thus, we find that for this model also $a(t)$, $\rho(t)$ and $\Lambda(t)$ follow the same relationship with time as in the barotropic case [10] and γ is related to matter and vacuum energy densities by the same relation as in the barotropic case [10].

2. Non-dust case ($\omega \neq 0$)

When $\omega \neq 0$, then $D \neq 0$. In this case, following the same procedure as in the $\Lambda \sim (\dot{a}/a)^2$ model, it is easy to show that no real solution is possible for $\omega > 0$. For $\omega = -1$, we have

$$a(t) = C_6 \exp \left[\frac{3}{8\pi G(\gamma + 1)} \right]^{1/2} t \quad (41)$$

where C_6 is a constant.

$$H = \left[\frac{3}{8\pi G(\gamma + 1)} \right]^{1/2}, \quad (42)$$

$$\rho(t) = \left[\frac{3}{8\pi G(\gamma + 1)} \right]^2, \quad (43)$$

$$\Lambda(t) = \frac{9\gamma}{8\pi G(\gamma + 1)^2}. \quad (44)$$

Here we find that in non-dust case, solutions are possible for vacuum fluid ($\omega = -1$), quintessence ($-1 < \omega < 0$) and phantom energy ($\omega < -1$). For vacuum fluid, solution is independent of n whereas for quintessence and phantom energy solution depends on n . In this case also, γ is found to be the ratio of vacuum energy density and matter energy density. For physical reality, $\gamma > -1$ to be imposed on the solutions.

C. $\Lambda \sim \ddot{a}/a$

If we use the supposition $\Lambda = \beta(\ddot{a}/a) = \beta(\dot{H} + H^2)$, then from equation (3) we get, $3H^2 = 8\pi G\rho + \beta(\dot{H} + H^2)$. Therefore

$$\rho = \frac{1}{8\pi G} [(3 - \beta)H^2 - \beta\dot{H}]. \quad (45)$$

Then, using (5), from (4) we have, $3(\dot{H} + H^2) = -12\pi G\omega\rho^n - 4\pi G\rho + \beta(\dot{H} + H^2)$ which on simplification becomes

$$\frac{3}{2} [(3 - \beta)H^2 + (2 - \beta)\dot{H}] = -\frac{12\pi G\omega}{(8\pi G)^n} [(3 - \beta)H^2 - \beta\dot{H}]^n. \quad (46)$$

For simplifying equation (46), let us investigate in some particular cases. If we specifically choose $n = 0$ (i.e. $p = \omega$), then equation (46) reduces to

$$(2 - \beta)\dot{H} + (3 - \beta)H^2 = -8\pi G\omega. \quad (47)$$

1. Dust case ($\omega = 0$)

For pressureless dust ($\omega = 0$) equation (47) becomes

$$(2 - \beta)\frac{dH}{dt} + (3 - \beta)H^2 = 0. \quad (48)$$

By solving equation (48) and using the initial condition that $a = 0$ when $t = 0$ we get

$$\frac{da}{dt} = a \left(\frac{2 - \beta}{3 - \beta} \right) \frac{1}{t}. \quad (49)$$

Then from equation (49) we finally get our solution set as

$$a(t) = C_7 t^{(\beta-2)/(\beta-3)} \quad (50)$$

where C_7 is a constant.

$$\rho(t) = \frac{1}{4\pi G} \left(\frac{\beta - 2}{\beta - 3} \right) \frac{1}{t^2}, \quad (51)$$

$$\Lambda(t) = \frac{\beta(\beta - 2)}{(\beta - 3)^2} \frac{1}{t^2}, \quad (52)$$

$$H(t) = \left(\frac{\beta - 2}{\beta - 3} \right) \frac{1}{t}. \quad (53)$$

Thus, for $n = 0$, the solution set is same for $a(t)$, $\rho(t)$ and $\Lambda(t)$ as obtained by Ray and Mukhopadhyay [10] in the dust case with barotropic equation of state.

2. Non-dust case ($\omega \neq 0$)

In non-dust case, equation (47) can be written as

$$(2 - \beta)\frac{dH}{dt} + (3 - \beta)H^2 = -8\pi G\omega. \quad (54)$$

By solving equation (47) and remembering that $H = 0$ when $t = 0$ we get

$$\frac{da}{dt} = \nu a \tan(\mu t) \quad (55)$$

where $\mu = [8\pi G\omega(3 - \beta)/(\beta - 2)]^{1/2}$ and $\nu = [8\pi G\omega/(3 - \beta)]^{1/2}$.

After solving equation (55) we have

$$a(t) = C_8 \left[\sec \frac{\{8\pi G\omega(3-\beta)\}^{1/2}}{(\beta-2)} t \right]^{\{8\pi G\omega/(3-\beta)\}^{1/2}}, \quad (56)$$

$$\rho(t) = \frac{(8\pi G\omega)^{3/2}(3-\beta)^{1/2}}{8\pi G(\beta-2)^2} \left[\{8\pi G\omega(3-\beta)\}^{1/2} \tan^2 \frac{\{8\pi G\omega(3-\beta)\}^{1/2}}{(\beta-2)} t - \beta \sec^2 \frac{\{8\pi G\omega(3-\beta)\}^{1/2}}{(\beta-2)} t \right], \quad (57)$$

$$\Lambda(t) = \frac{(8\pi G\omega)^{3/2}}{(\beta-2)^2} \beta \left[(3-\beta)^{1/2} \sec^2 \frac{\{8\pi G\omega(3-\beta)\}^{1/2}}{(\beta-2)} t + (8\pi G\omega)^{1/2} \tan^2 \frac{\{8\pi G\omega(3-\beta)\}^{1/2}}{(\beta-2)} t \right], \quad (58)$$

$$H(t) = \frac{8\pi G\omega}{(\beta-2)} \tan \frac{\{8\pi G\omega(3-\beta)\}^{1/2}}{(\beta-2)} t. \quad (59)$$

From the above solution set, it is clear that if $\omega > 0$, then for real $a(t)$, $\rho(t)$, $\Lambda(t)$ and $H(t)$, β must be less than 3. For negative ω (i.e., for vacuum fluid, quintessence and phantom energy) real values of $a(t)$ and $H(t)$ can be obtained, but $\Lambda(t)$ and $\rho(t)$ become imaginary. If $(\beta-3) = 0$, then $\rho(t)$, $\Lambda(t)$ and $H(t)$ are zero but $a(t)$ is undefined. Since \tan and \sec are both increasing functions of t , then $a(t)$, $\Lambda(t)$ and $H(t)$ increase with time. Increasing $a(t)$ supports the idea of an accelerating Universe, but increasing $\Lambda(t)$ and $H(t)$ are contrary to the present status of those two parameters.

For $t = 0$, we get respectively from equations (56) - (59) the following physical parameters:

$$a(t) = C_8, \quad (60)$$

$$\rho(t) = -\beta \frac{1}{8\pi G} \frac{(8\pi G\omega)^{3/2}(3-\beta)^{1/2}}{(\beta-2)^2}, \quad (61)$$

$$\Lambda(t) = \frac{(8\pi G\omega)^{3/2}}{(\beta-2)^2} \beta (3-\beta)^{1/2}, \quad (62)$$

$$H(t) = 0. \quad (63)$$

Since $a(t)$ assumes a constant value for $t = 0$, then our model hints at the existence of some kind of quantum fluctuation at the time of Big Bang. Now, equation (61) tells us that for physically valid ρ , $\omega > 0$ and $\beta < 0$. For a negative β we get an attractive Λ . So, as a whole this model also presents some interesting as well as awkward cosmological picture.

IV. DISCUSSION

Present investigation reveals that for pressureless dust, all the three models reflect the same result as obtained by Ray and Mukhopadhyay [10] in dust case with barotropic equation of state. But, in non-dust cases (i.e., $\omega \neq 0$), $\Lambda \sim (\dot{a}/a)^2$ and $\Lambda \sim \rho$ models show exponential expansion of the Universe for negative ω whereas no real situation is possible for $\omega > 0$. Thus, we can say that for dust-filled Universe, there is no distinction between barotropic and polytropic equations of state. This is not at all unexpected because for $\omega = 0$, we have $p = 0$, whatever may be the value of n in the equation of state. Also, non-dust cases for all the three models present us some unpleasant results in terms of constant H and ρ . These situations can be explained if we assume that the non-dust cases reflect the picture of early Universe when inflation occurred due to the presence of quintessence ($-1 < \omega < 0$) or vacuum fluid ($\omega = -1$) or phantom energy ($\omega < -1$). So, using polytropic equation of state it has been possible to show that non-dust cases admit the presence of a driving force behind inflation in the form of either quintessence or vacuum fluid or phantom energy and in the dust cases there is no distinction between different equation of states. Moreover, present models do not depend on any particular value of n in the polytropic equation of state. This proves the generality of the present investigation.

Acknowledgments

One of the authors (SR) would like to express his gratitude to the authority of IUCAA, Pune for providing him the Associateship Programme under which a part of this

work was carried out.

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